

HESSENBERG VARIETIES OF PARABOLIC TYPE

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ABSTRACT. This paper proves a combinatorial relationship between two well-studied subvarieties of the flag variety: certain Hessenberg varieties, which are a family of subvarieties of the flag variety that includes Springer fibers, and Schubert varieties, which induce a well-known basis for the cohomology of the flag variety. The main result shows that the Betti numbers of parabolic Hessenberg varieties decompose into a combination of those of Springer fibers and Schubert varieties associated to the parabolic. As a corollary we show that the Betti numbers of some parabolic Hessenberg varieties in Lie type A are equal to those of a specific union of Schubert varieties. The corollary uses (and generalizes) recent work of the same authors that proves the analogous result for certain Springer fibers in Lie type A .

1. INTRODUCTION

This paper analyzes the structure of parabolic Hessenberg varieties. Our main result shows that in all Lie types, the Betti numbers of parabolic Hessenberg varieties decompose into a combination of those of Springer fibers and Schubert varieties associated to the parabolic. As an application, we show that in Lie type A the Betti numbers of parabolic Hessenberg varieties for three-row or two-column nilpotent operators are equal to the Betti numbers of a specific union of Schubert varieties.

Schubert varieties are well-understood subvarieties of the flag variety G/B whose geometry is intrinsically connected to the combinatorics of the Weyl group. The Schubert cells $C_w = BwB/B$ form a CW-decomposition of the flag variety. The closure relations between these cells are determined by the Bruhat order on W and the dimension of the corresponding Schubert variety \overline{C}_w is given by the Bruhat length of w , which is the number of inversions of w in type A . The cohomology classes of Schubert varieties are given by the Schubert polynomials of Bernstein-Gelfand-Gelfand [BGG], which form an important basis for the cohomology of the flag variety. The study of Schubert varieties and Schubert polynomials fundamentally relates results in geometry, combinatorics, Lie theory, and representation theory; see [BP], [CK], and [Ku] for just a few examples. (For more, see surveys like Fulton's [F] or Billey-Lakshmibai's [BL].)

Hessenberg varieties are a family of subvarieties of the flag variety parametrized by an element of the Lie algebra and a Hessenberg space. We describe them here for Lie type A and give a general definition in Section 2. In Lie type A the flag gB can be written as a collection of nested subspaces $V_\bullet = (\{0\} \subseteq V_1 \subseteq \cdots \subseteq V_{n-1} \subseteq V)$ where each V_i is an i -dimensional subspace of a fixed complex n -dimensional vector space V . An element X of the Lie algebra is an $n \times n$ matrix. A Hessenberg space is determined by what's called a Hessenberg function: a map $h : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ satisfying both $h(i) \geq i$ for $i = 1, 2, \dots, n$ and $h(i) \geq h(i-1)$ for $i = 2, \dots, n$. Given any choice of X and h , the flag V_\bullet is an element of the Hessenberg variety $\mathcal{B}(X, h)$ if $X(V_i) \subseteq V_{h(i)}$ for all $i = 1, 2, \dots, n$.

The homology and cohomology of Hessenberg varieties arises naturally in many different contexts. Hessenberg varieties were first defined by De Mari, Procesi, and Shayman because of applications to numerical analysis [dMPS]. When $h(i) = i+1$ for all $i = 1, 2, \dots, n-1$ and X is a regular

nilpotent element (namely it has just one Jordan block), the variety $\mathcal{B}(X, h)$ is called the Peterson variety. Peterson and Kostant used Peterson varieties to construct the quantum cohomology of the flag variety [K2]. When $h(i) = i$ for all $i = 1, \dots, n$ and X is nilpotent, the corresponding Hessenberg variety is the Springer fiber \mathcal{B}^X and is the set of all flags stabilized by X . Springer proved that the cohomology of the Springer fibers admits geometric representations of S_n (or Weyl groups in arbitrary Lie type) [Sp, Sp2]. When X is regular and semisimple (namely has n distinct eigenvalues), the equivariant cohomology of the corresponding Hessenberg variety admits an action of the symmetric group [T2]. Recent work of Shareshian and Wachs conjectured this same representation appears via certain quasisymmetric functions [SW]; the conjecture was proven by Brosnan and Chow [BC] and by Guay-Paquet [G] using different methods.

The conjecture that the parameters X and h determine a union of Schubert varieties with the same homology as $\mathcal{B}(X, h)$ is a key step in one construction of the equivariant cohomology of Hessenberg varieties. This conjecture has been proven in special cases. Harada and the second author proved that the Betti numbers of Peterson varieties are the same as the Betti numbers of the Schubert variety corresponding to $s_1 s_2 \cdots s_n$ and used that to construct their equivariant cohomology [HT]. More generally when X is a regular nilpotent element in Lie type A , the Betti numbers of the corresponding regular nilpotent Hessenberg variety were described by Mbirika in [M]; Reiner noted that these Betti numbers agree with the Betti numbers of a kind of Schubert variety called a Ding variety [D, DMR]. Reiner further conjectured that the cohomology rings of these varieties are isomorphic, though this has just been disproved by Abe, Harada, Horiguchi, and Masuda's construction of the equivariant cohomology of regular nilpotent Hessenberg varieties [AHHM]. The authors of the current paper recently showed that the Betti numbers of Springer fibers corresponding to partitions with at most three rows or two columns coincide with the Betti numbers of a specific union of Schubert varieties [PT].

In this paper, we consider parabolic Hessenberg spaces in $\mathfrak{gl}_n(\mathbb{C})$, namely those matrices in a fixed block upper triangular form. The sizes of the blocks in the parabolic Hessenberg space are indexed by a partition $\mu = (m_1, m_2, \dots, m_k)$ of n . A parabolic Hessenberg function can be defined from the corresponding parabolic Hessenberg space, but it is also a Hessenberg function $h : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ whose image consists precisely of those integers i that are fixed by h . Let J be the set of integers i that are not fixed by h and let $W_J = \langle s_i : i \in J, i \leq n-1 \rangle$ be generated by simple transpositions $s_i = (i, i+1)$. We use the classical fact that each permutation $w \in S_n$ can be written uniquely as $w = vy$ where v is a minimal-length coset representative for vy in W/W_J and $y \in W_J$ [BB].

Our main result proves that the Betti numbers of each parabolic Hessenberg variety are a sum of the Poincaré polynomials for the corresponding Springer variety, shifted by the Betti numbers of certain Schubert varieties (see Corollary 3.7 for details).

As an application, our second result builds on this in the case when the conjugacy class of X corresponds to a partition with at most three rows or two columns, describing the Betti numbers of each parabolic Hessenberg variety in terms of a specific union of Schubert varieties. Our proof relies on earlier results for Springer fibers that associate a permutation v_T called a Schubert point to each permutation flag $vB \in \mathcal{B}^X$. The following is proven in Section 4 (see Theorem 4.10).

Theorem 1. *Let $X \in \mathfrak{gl}_n(\mathbb{C})$ be a nilpotent matrix with Jordan type corresponding to a partition with at most three rows or two columns, and fix a parabolic Hessenberg function with fixed points J . Let*

$$W(X, J) = \left\{ v \in W : \begin{array}{l} v \text{ is a minimal-length coset representative} \\ \text{for } W/W_J \text{ and the flag } vB \text{ is in } \mathcal{B}^X \end{array} \right\}$$

and for each $v \in W(X, J)$ let v_T denote the corresponding Schubert point. Then the following homology groups are isomorphic:

$$H_*(\mathcal{B}(X, h)) = H_*(\cup_{v \in W(X, J)} \overline{C}_{v_T w_J})$$

where w_J denotes the longest word in W_J (and homology is taken with rational coefficients).

To prove this result, we show that the structure of the parabolic Hessenberg variety $\mathcal{B}(X, h)$ is in large part determined by the structure of the Springer fiber \mathcal{B}^X together with the flag variety of the Levi subgroup determined by J . Like the methods in other papers working on this conjecture, the arguments in this paper do not appear to extend easily to the more general setting of arbitrary Hessenberg spaces.

Our description of the geometry of parabolic Hessenberg varieties concludes with a brief analysis of their irreducible components in Section 5. We partially describe the closure relations among the intersections $C_w \cap \mathcal{B}(X, h)$ and conclude with an open question which asks for a combinatorial description of the irreducible components in terms of row-strict tableaux. A full understanding of the closure relations between these cells is unknown even for Springer varieties, despite over forty years' worth of work on them.

Although our methods are combinatorial and Lie theoretic, the fact that the Betti numbers of Hessenberg varieties are equal to those of a union of Schubert varieties in all of these cases indicates some deeper geometric phenomenon is in play. We know Hessenberg varieties are not simply unions of Schubert varieties because they exhibit different singularities. We conjecture there is some degeneration from nilpotent Hessenberg varieties (perhaps with restrictions on the Jordan type of the nilpotent operator) to a union of Schubert varieties, similar to the degeneration given by Knutson and Miller from a Schubert variety to a collection of line bundles [KM]. However, the geometry of nilpotent Hessenberg varieties is much less well understood than that of the Schubert varieties suggesting that some new methods will be necessary to find such a degeneration.

This paper is structured as follows. The second section covers background information and notation as well as a proposition which allows us to merge results from [T] and [P]. The third analyzes the structure of parabolic Hessenberg varieties. All the results in Section 3, including our main result, hold for Hessenberg varieties corresponding to any complex algebraic reductive group. The fourth section specializes to Lie type A in which our underlying algebraic group is $G = GL_n(\mathbb{C})$. Section 4 proves Theorem 1 using the combinatorics of row-strict tableaux. Section 5 concludes with a partial description of the irreducible components of parabolic Hessenberg varieties.

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2. PRELIMINARIES

This section establishes key definitions, as well as some results that restate past work in a form that will be useful in what follows.

We use the following notation:

- G is a complex algebraic reductive group with Lie algebra \mathfrak{g} .
- B is a fixed Borel subgroup of G with Lie algebra \mathfrak{b} .
- U is the maximal unipotent subgroup of B with Lie algebra \mathfrak{u} .
- $T \subset B$ is a fixed maximal torus with Lie algebra \mathfrak{t} .
- $W = N_G(T)/T$ denotes the Weyl group.
- We fix a representative $w \in N_G(T)$ for each $w \in W$ and use the same letter for both.

- Φ^+ , Φ^- , and Δ are the positive, negative and simple roots associated to the previous data.
- Given $\gamma \in \Phi$ we write \mathfrak{g}_γ for the root space in \mathfrak{g} corresponding to γ and fix a generating root vector $E_\gamma \in \mathfrak{g}_\gamma$.
- We denote by s_γ the reflection in W corresponding to $\gamma \in \Phi$ and write $s_{\alpha_i} = s_i$ when $\alpha_i \in \Delta$.

After Section 3 we specialize to the case when $G = GL_n(\mathbb{C})$ is the group of $n \times n$ invertible matrices and $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ is the collection of $n \times n$ matrices. This is also our main example throughout. In this setting B is the subgroup of invertible upper triangular matrices, T is the diagonal subgroup, and $W \cong S_n$ can be described as the symmetric group on n letters. The positive roots are

$$\Phi^+ = \{\alpha_i + \alpha_{i+1} \cdots + \alpha_{j-1} : 1 \leq i < j \leq n\}$$

where $\alpha_i = \epsilon_i - \epsilon_{i-1}$ and $\epsilon_i(X) = X_{ii}$ for all $X \in \mathfrak{gl}_n(\mathbb{C})$. Let E_{ij} denote the elementary matrix with 1 in the (i, j) -entry and 0 in every other entry. The root vector corresponding to the root $\gamma = \alpha_i + \alpha_{i+1} \cdots + \alpha_{j-1}$ for each $1 \leq i < j \leq n$ is $E_\gamma = E_{ij}$. When working in the type A setting we will identify (i, j) with the root $\alpha_i + \alpha_{i+1} \cdots + \alpha_{j-1}$ whenever it is notationally convenient. For $\gamma_1, \gamma_2 \in \Phi$ we write $\gamma_1 \geq \gamma_2$ if $\gamma_1 - \gamma_2$ is a sum of positive roots or $\gamma_1 = \gamma_2$.

Definition 2.1. *The inversion set of the Weyl group element w is the set*

$$N(w) = \{\gamma \in \Phi^+ : w(\gamma) \in \Phi^-\}$$

This generalizes to arbitrary Lie type the classical definition of an inversion, in which the pair (i, j) is an inversion of $w \in S_n$ if $i < j$ and $w(i) > w(j)$. If we identify the pair (i, j) with the root $\alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} \in \Phi^+$ then (i, j) is an inversion of w in the classical sense if and only if $\alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} \in N(w)$. Note that if $\ell(w)$ denotes the (Bruhat) length function on W then $\ell(w) = |N(w)|$.

The projective variety $\mathcal{B} = G/B$ is called the flag variety. When $G = GL_n(\mathbb{C})$ the flag variety can be identified with the set of full flags $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_{n-1} \subseteq V$ in a complex n -dimensional vector space. The main focus of this paper is a collection of subvarieties of the flag variety called Hessenberg varieties, which we now define.

Definition 2.2. *A linear subspace $H \subseteq \mathfrak{g}$ is a Hessenberg space if $\mathfrak{b} \subseteq H$ and $[\mathfrak{b}, H] \subseteq H$.*

The condition that $[\mathfrak{b}, H] \subseteq H$ implies this subspace of \mathfrak{g} can be written as

$$\mathfrak{t} \oplus \bigoplus_{\gamma \in \Phi_H} \mathfrak{g}_\gamma$$

over an index set $\Phi_H \subset \Phi$ determined by (and determining) H . Let $\Phi_H^- = \Phi_H \cap \Phi^-$ denote the negative roots in this index set. In type A the set of indices Φ_H forms a “staircase” shape, in the sense that if (i, j) is in Φ_H then so are all (k, j) with $1 \leq k \leq i$ and all (i, k) with $j \leq k \leq n$. In other words if matrices in H are not identically zero in the entry (i, j) , then they can be nonzero in any entry above or to the right of (i, j) . Alternatively the Hessenberg space $H \subseteq \mathfrak{gl}_n(\mathbb{C})$ is uniquely associated to a Hessenberg function $h : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ (as defined in the Introduction) by the rule that $h(i)$ equals the number of entries that are not identically zero in the i -th column of H . The condition that $h(i) \geq i$ is equivalent to the requirement that $\mathfrak{b} \subseteq H$ while the condition $h(i) \geq h(i-1)$ is equivalent to the requirement $[\mathfrak{b}, H] \subseteq H$.

Example 2.3. *As an example, we give a Hessenberg function h and the corresponding Hessenberg space H when $n = 5$. The space of matrices H is described by indicating where the zeroes must be*

in each matrix; the entries designated $*$ can be filled freely with any element of \mathbb{C} .

$$H = \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \end{pmatrix} \longleftrightarrow h(i) = \begin{cases} 2 & \text{if } i = 1, 2 \\ 4 & \text{if } i = 3 \\ 5 & \text{if } i = 4, 5 \end{cases}$$

Hessenberg varieties are parametrized by two objects: a Hessenberg space $H \subset \mathfrak{g}$ and an element $X \in \mathfrak{g}$ as follows.

Definition 2.4. Fix a Hessenberg space $H \subset \mathfrak{g}$ and an element $X \in \mathfrak{g}$. The Hessenberg variety is the subvariety of the flag variety given by

$$\mathcal{B}(X, H) = \{gB \in G/B : g^{-1} \cdot X \in H\}$$

where $g \cdot X := \text{Ad}(g)X = gXg^{-1}$.

A key example is the case in which $H = \mathfrak{b}$. Then $\mathcal{B}(X, \mathfrak{b})$ consists of all flags gB such that $g^{-1} \cdot X \in \mathfrak{b}$ or equivalently $X \in g \cdot \mathfrak{b}$. This is the set of flags corresponding to Borel subalgebras containing X , is called the Springer fiber, and denoted by \mathcal{B}^X .

Hessenberg varieties have an affine paving, which is like a CW-complex structure but with less restrictive closure conditions.

Definition 2.5. A paving of an algebraic variety Y is a filtration by closed subvarieties

$$Y_0 \subset Y_1 \subset \cdots \subset Y_i \subset \cdots \subset Y_d = Y.$$

A paving is affine if every $Y_i - Y_{i-1}$ is a finite disjoint union of affine spaces.

Like CW-complexes, affine pavings can be used to compute the Betti numbers of a variety. A reference for the following Lemma is [F2, 19.1.1].

Lemma 2.6. Let Y be an algebraic variety with an affine paving

$$Y_0 \subset Y_1 \subset \cdots \subset Y_i \subset \cdots \subset Y_d = Y$$

and let n_k denote the number of affine components of dimension k , or zero if n_k is zero. Then the compactly-supported cohomology groups of Y are given by

$$H_c^{2k}(Y) = \mathbb{Z}^{n_k}$$

The Bruhat decomposition of the flag variety induces a well-known paving by affines [BL, Section 2.6]. Decompose the flag variety as $\mathcal{B} = \bigsqcup_{w \in W} C_w$ where $C_w = BwB/B$ is the Schubert cell indexed by $w \in W$. The paving of \mathcal{B} is given by

$$\mathcal{B}_i = \bigsqcup_{\ell(w)=i} \overline{C}_w$$

is affine because $\overline{C}_w = \bigsqcup_{y \leq w} C_y$ where \leq denotes the Bruhat order and because each $C_w \cong \mathbb{C}^{\ell(w)}$.

Calculating the Poincaré polynomial of a Schubert variety or a union of Schubert varieties is a application of this combinatorial description, as shown in the following example.

Example 2.7. Let $G = GL_4(\mathbb{C})$ and consider $w = s_1 s_2 s_3 s_1$. The set

$$\{v \in W \text{ such that } v \leq w\}$$

is the set of all possible subwords of w . When $w = s_1 s_2 s_3 s_1$ this set is

$$\{s_1 s_2 s_3 s_1, s_1 s_2 s_3, s_1 s_2 s_1, s_2 s_3 s_1, s_1 s_2, s_2 s_1, s_2 s_3, s_1 s_3, s_1, s_2, s_3, e\}.$$

Therefore the Poincaré polynomial of \overline{C}_w is $P(\overline{C}_w, t) = 1 + 3t + 4t^2 + 3t^3 + t^4$.

Intersecting the Hessenberg variety $\mathcal{B}(X, H)$ with certain choices of Schubert cells gives an affine paving of $\mathcal{B}(X, H)$. These intersections are called Hessenberg Schubert cells. We now describe the Hessenberg Schubert cells that we use in this paper. We begin with an observation that allows us to simplify calculations without loss of generality.

Remark 2.8 ([T], Proposition 2.7). *The Hessenberg varieties corresponding to X and $g \cdot X$ are homeomorphic.*

Let $X \in \mathfrak{g}$ be nilpotent and fix H . The previous remark says that we can choose X within its conjugacy class to make computations as convenient as possible. We will assume that $X = \sum_{\gamma \in \Phi_X} E_\gamma$ for a subset $\Phi_X \subseteq \Phi^+$ and that

$$(2.9) \quad U \cdot X = X + \mathcal{V}$$

where $\mathcal{V} \subset \mathfrak{u}$ is a nilpotent ideal of \mathfrak{b} such that $\mathcal{V} = \bigoplus_{\gamma \in \Phi(\mathcal{V})} \mathfrak{g}_\gamma$ and $\Phi_X \cap \Phi(\mathcal{V}) = \emptyset$. Equivalently these two conditions can be rephrased as

$$\Phi(\mathcal{V}) = \{\gamma \in \Phi^+ : \gamma \succeq \alpha \text{ for some } \alpha \in \Phi_X\}$$

To demonstrate the meaning of Equation (2.9), consider the following example.

Example 2.10. *Each of the following matrices is an element of the conjugacy class $\mathcal{C}_{(2,2)}$ in $\mathfrak{gl}_4(\mathbb{C})$.*

$$X_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Indeed, X_1 is in Jordan form and $X_2 = s_2 \cdot X_1$. Consider the U -orbit of each of these matrices. Every element of U is of the form $I + \sum_{\gamma \in \Phi^+} x_\gamma E_\gamma$ for some $x_\gamma \in \mathbb{C}$. In particular, if $u = I + x_{23}E_{23}$ for $t \in \mathbb{C}$, then

$$u \cdot X_1 = \begin{pmatrix} 0 & 1 & -x_{23} & 0 \\ 0 & 0 & 0 & x_{23} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = X_1 + x_{23}(E_{24} - E_{13}).$$

This shows that

$$X_1 + x_{23}(E_{24} - E_{13}) \in U \cdot X_1.$$

However, computing $u \cdot X_1$ for other elements of U shows that $X_1 + Y \notin U \cdot X_1$ for either $Y \in \mathfrak{g}_{\alpha_1 + \alpha_2}$ or $Y \in \mathfrak{g}_{\alpha_2 + \alpha_3}$. Therefore, X_1 is a nilpotent element for which Equation (2.9) does not hold. On the other hand, $U \cdot X_2 = X_2 + \mathfrak{g}_{\alpha_1 + \alpha_2 + \alpha_3}$ which can be calculated explicitly using either $u = I + x_{13}E_{13}$ or $u = I + x_{24}E_{24}$, for nonzero $x_{13}, x_{24} \in \mathbb{C}$.

The condition given in Equation (2.9) can be satisfied for all nilpotent conjugacy classes in type A and for many in other types. More precisely whenever a nilpotent is regular in a Levi subalgebra of \mathfrak{g} , there exists another element of the same nilpotent conjugacy class that satisfies this condition [P]. If $X \in \mathfrak{gl}_n(\mathbb{C})$ then X can be conjugated into Jordan form, which implies that every nilpotent element in type A is regular in a Levi.

The next lemma describes the Hessenberg Schubert cells for this choice of X . It restates results from the literature in ways that are useful later.

Lemma 2.11. *Fix a Hessenberg space H . Let $X \in \mathfrak{g}$ be a nilpotent element such that the U -orbit through X can be calculated as in Equation (2.9). Then the intersection $C_w \cap \mathcal{B}(X, H)$ is nonempty if and only if $w^{-1} \cdot X \in H$. If nonempty then $C_w \cap \mathcal{B}(X, H) \cong \mathbb{C}^{d_w}$ where*

$$d_w = |N(w^{-1}) \cap \Phi(\mathcal{V})^c| + |N(w^{-1}) \cap \Phi(\mathcal{V}) \cap w(\Phi_H^-)|.$$

Proof. We know that $C_w \cap \mathcal{B}(X, H) \neq \emptyset$ if and only if $w^{-1} \cdot X \in H$ and that when nonempty $C_w \cap \mathcal{B}(X, H) \cong \mathbb{C}^{d_w}$ by [T, Theorem 6.1 and Corollary 6.3] in type A and [P, Proposition 3.7] for other types. We have only to show our dimension assertion, which follows from the identity

$$d_w = |N(w^{-1})| - \dim \mathcal{V} / (\mathcal{V} \cap w \cdot H)$$

given in [P, Proposition 3.7]. Expanding and then simplifying this formula gives

$$\begin{aligned} d_w &= |N(w^{-1})| - |\{\gamma \in \Phi(\mathcal{V}) : w^{-1}(\gamma) \notin \Phi_H\}| \\ &= |N(w^{-1})| - |\{\gamma \in N(w^{-1}) \cap \Phi(\mathcal{V}) : w^{-1}(\gamma) \in \Phi^- - \Phi_H^-\}| \\ &= |N(w^{-1}) \cap \Phi(\mathcal{V})^c| + |\{\gamma \in \Phi(\mathcal{V}) \cap N(w^{-1}) : w^{-1}(\gamma) \in \Phi_H^-\}| \end{aligned}$$

which simplifies to the desired formula. \square

In type A we can write a matrix X satisfying the condition from Equation (2.9) explicitly (and will use this construction later). We start with some notation, following [T, Definitions 4.1 and 4.2].

Definition 2.12. *Let X be an upper-triangular nilpotent $n \times n$ matrix. Define a function $\text{piv} : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ as follows. For each i , define $\text{piv}(i) = 0$ if no pivot occurs in the i -th column of X and $\text{piv}(i) = j$ if a pivot occurs in row j of the i -th column of X . We say that X is in highest form if piv is an increasing function.*

For instance consider the matrix X in Example 2.14. The images of the integers under piv are $0, 0, 1, 2, 4$ so X is in highest form.

Recall that the conjugacy classes of nilpotent matrices in $\mathfrak{gl}_n(\mathbb{C})$ are determined by the sizes of their Jordan blocks. Let λ be a partition of n associated to a fixed Jordan type. We construct a highest-form representative for the matrices of Jordan type λ as in [T, §4].

Definition 2.13. *Fill the boxes of λ with integers 1 to n starting at the bottom of the leftmost column and moving up the column by increments of one. Then move to the lowest box of the next column and so on. This filling is called the base filling of λ . Let X be the matrix so that $X_{kj} = 1$ if j fills a box directly to the right of k and $X_{kj} = 0$ otherwise. Then X is in highest form.*

Example 2.14. *Let $n = 5$ and $\lambda = (3, 2)$. Definition 2.13 gives the following base filling of λ and representative X of Jordan type λ .*

$$\begin{array}{|c|c|c|} \hline 2 & 4 & 5 \\ \hline 1 & 3 & \\ \hline \end{array} \quad \text{and} \quad X = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We can write X as a linear combination of elementary matrices. For each $1 \leq k \leq n$ let r_k denote the entry filling the box directly to the right of k and set $r_k = 0$ if there is no such box. Then $X = \sum_{\{k:r_k \neq 0\}} E_{kr_k}$. In the notation of roots we have

$$\Phi_X = \{\alpha_k + \alpha_{k+1} \cdots + \alpha_{r_k-1} : 1 \leq k \leq n \text{ and } r_k \neq 0\}.$$

In fact X also satisfies the conditions of Equation (2.9). Our proof depends on the fact that X is in highest form and so its pivots are in columns as far to the right as possible. In other words, the following proposition shows that the highest form matrices defined combinatorially by the second author [T] actually satisfy the algebraic conditions given by the first author [P] in slightly different constructions of pavings for nilpotent Hessenberg varieties.

Proposition 2.15. *Suppose $G = GL_n(\mathbb{C})$. Let $X \in \mathfrak{gl}_n(\mathbb{C})$ be an upper-triangular nilpotent matrix of Jordan type λ such that X is in highest form with entries given by Definition 2.13. Then $U \cdot X = X + \mathcal{V}$ where $\mathcal{V} = \bigoplus_{\gamma \in \Phi(\mathcal{V})} \mathfrak{g}_\gamma$ satisfies*

$$\Phi(\mathcal{V}) = \left\{ \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} : \begin{array}{l} \bullet 1 \leq i < j \leq n \\ \bullet j \geq r_i \\ \bullet j \text{ fills a box in a column of the} \\ \text{base filling of } \lambda \text{ to the right of } i \end{array} \right\}.$$

Proof. The set $\Phi(\mathcal{V})$ given in the statement of the proposition can also be described as the roots $\gamma \in \Phi^+$ such that $\gamma \succeq \gamma'$ for some $\gamma' \in \Phi_X$. Therefore, we certainly have that $U \cdot X \subseteq X + \mathcal{V}$ for \mathcal{V} as described above.

To show equality, it is enough to show that $\dim(U \cdot X) = \dim(\mathcal{V})$. Indeed, since the U -orbit $U \cdot X$ is closed (see [H], Exercise 17.8), if $\dim(U \cdot X) = \dim(\mathcal{V})$, then $U \cdot X = X + \mathcal{V}$. Recall that $\dim(U \cdot X) = \dim(U) - \dim(Z_U(X))$ where $Z_U(X)$ denotes the centralizer in U of X . The group U is unipotent and therefore is diffeomorphic to \mathfrak{u} via the matrix exponential $\exp : \mathfrak{g} \rightarrow G$. Thus we instead consider $\dim(\mathfrak{u}) - \dim(\mathfrak{z}_{\mathfrak{u}}(X))$ where $\mathfrak{z}_{\mathfrak{u}}(X) = \{Y \in \mathfrak{u} : [Y, X] = 0\}$. To show that

$$\dim(\mathfrak{u}) - \dim(\mathfrak{z}_{\mathfrak{u}}(X)) = \dim(\mathcal{V}) = |\Phi(\mathcal{V})|$$

we will show that for each root vector $E_\gamma \in \mathcal{V}$ there exists a $Y \in \mathfrak{u}$ such that $[Y, X] = E_\gamma$. It follows that the linear map $ad_X : \mathfrak{u} \rightarrow \mathcal{V}$ given by $Y \mapsto [Y, X]$ is onto, so $\dim(\mathfrak{u}) - \dim(\mathfrak{z}_{\mathfrak{u}}(X)) = \dim(\mathcal{V})$ as desired.

Let $E_\gamma = E_{ij} \in \mathcal{V}$ so that $\gamma = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} \in \Phi(\mathcal{V})$. By assumption, j fills a box in λ to right of i but not the box directly to the right of i or below that entry by the assumption that $j \geq r_i$. Given any such j , let j' denote the entry of the box directly to the left of j , so $r_{j'} = j$.

Proceed by induction on i . If $i = 1$ set $Y = E_{1j'}$. Note that $r_k \neq 1$ for all k since 1 will always fill the lowest box of the first column in the base filling of λ and therefore cannot fill the box directly to the right of k . We see that

$$[Y, X] = [E_{1j'}, \sum_{\{k:r_k \neq 0\}} E_{kr_k}] = [E_{1j'}, E_{j'r_{j'}} + \sum_{\{k \neq j': r_k \neq 0\}} E_{kr_k}] = E_{1j} - \delta_{1r_k} E_{kj'} = E_{1j} = E_\gamma$$

so $Y = E_{1j'}$ has the desired property.

Now consider $\gamma = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$ with $i > 1$. Since j does not occur directly to the right or below the entry directly to the right of i , j' occurs in the same column and above i or in a column to the right of i , implying $j' > i$ so $\gamma' = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j'-1} \in \Phi^+$. Set $Y' = E_{\gamma'} = E_{ij'}$. Then

$$[Y', X] = [E_{ij'}, \sum_{\{k:r_k \neq 0\}} E_{kr_k}] = [E_{ij'}, E_{j'r_{j'}} + \sum_{\{k \neq j': r_k \neq 0\}} E_{kr_k}] = E_{ij} - \sum_{\{k \neq j': r_k \neq 0\}} \delta_{ir_k} E_{kj'}.$$

If $\delta_{ir_k} = 0$ for all r_k appearing in this equation then $Y = Y'$ is the desired element of \mathfrak{u} . If not then there is exactly one k such that $i = r_k$, denote it by k' . In this case,

$$[Y', X] = E_{ij} - \sum_{k \neq j': r_k \neq 0} \delta_{ir_k} E_{kj'} = E_{ij} - E_{k'j'}.$$

Since $i = r_{k'}$, k' is the entry directly to the left of i in the base filling of λ and $k' < i$. By the induction hypothesis, there exists $Y'' \in \mathfrak{u}$ such that $[Y'', X] = E_{k'j'}$. Setting $Y = Y' + Y''$ gives $[Y, X] = E_{ij} = E_\gamma$ as desired. \square

3. PARABOLIC HESSENBERG VARIETIES

In this section we specialize to the case when the Hessenberg space H is a parabolic subalgebra. We begin with basic definitions and combinatorial facts and then prove the first of our main results, which shows how the combinatorics of parabolic Hessenberg varieties combine the combinatorics of the associated Springer fiber with that of the parabolic subgroup.

In type A a standard parabolic subalgebra consists of all matrices with a particular block upper triangular form. More generally a parabolic subalgebra is any subalgebra of \mathfrak{g} containing a Borel subalgebra and similarly for parabolic subgroups. This means that every parabolic subalgebra is a Hessenberg space. Moreover it is classically known that the subgroups of G that contain B are precisely the parabolic subgroups of the form

$$P_J = BW_JB = \bigsqcup_{w \in W_J} BwB$$

where $J \subseteq \Delta$ is a subset of simple roots and W_J is the subgroup of W generated by

$$S_J = \{s_i : \alpha_i \in J\}$$

[H, Theorem 29.3]. Note that $P_I \subseteq P_J$ if and only if $I \subseteq J$.

Write \mathfrak{p}_J for the corresponding parabolic subalgebra. **For the rest of the paper we assume $H = \mathfrak{p}_J$ for some $J \subseteq \Delta$.** We call the corresponding Hessenberg variety parabolic.

Let $\Phi_J \subseteq \Phi$ be the subsystem of roots spanned by J and denote its positive roots by Φ_J^+ and negative roots by Φ_J^- . The subalgebra \mathfrak{p}_J decomposes as

$$\mathfrak{p}_J = \mathfrak{m}_J \oplus \mathfrak{u}_J \text{ where } \mathfrak{m}_J = \mathfrak{t} \oplus \bigoplus_{\gamma \in \Phi_J} \mathfrak{g}_\gamma \text{ and } \mathfrak{u}_J = \bigoplus_{\gamma \in \Phi^+ - \Phi_J^+} \mathfrak{g}_\gamma$$

There is a corresponding decomposition of P into the semidirect product $M_J U_J$ where M_J and U_J are subgroups of G with $\text{Lie}(M_J) = \mathfrak{m}_J$ and $\text{Lie}(U_J) = \mathfrak{u}_J$. We will use the flag variety \mathcal{B}_J of the subgroup M_J in Corollary 3.7 and in the proof of the main theorem in Section 4.

Each coset in W/W_J contains a unique minimal-length representative. Denote the set of minimal-length representatives by W^J . This coset decomposition respects lengths: When $w \in W$ is written as $w = vy$ with $v \in W^J$ and $y \in W_J$ then $\ell(w) = \ell(v) + \ell(y)$ [BB, Proposition 2.4.4]. The set W^J can be characterized in different ways [K, Remark 5.13], which we now list.

Remark 3.1. Fix a Weyl group element v . The following statements are equivalent:

- (1) The Weyl group element v is in W^J .
- (2) Every positive root γ with $v^{-1}(\gamma) \in \Phi^-$ in fact satisfies $v^{-1}(\gamma) \in \Phi^- - \Phi_J^-$.
- (3) For all $\alpha_i \in J$ the simple root $\alpha_i \notin N(v)$.

In addition $y \in W_J$ normalizes $\Phi^+ - \Phi_J^+$ in the following sense.

Lemma 3.2. For all $y \in W_J$ we have

- $y(\Phi^+ - \Phi_J^+) = \Phi^+ - \Phi_J^+$ and
- $y(\Phi^- - \Phi_J^-) = \Phi^- - \Phi_J^-$.

We also use the following [K, Equation (5.13.2)] frequently, especially in the context of the decomposition $W = W^J W_J$.

Lemma 3.3. *Suppose that v and y are reduced words in W whose product $w = vy$ is also a reduced word. Then $\ell(w) = \ell(v) + \ell(y)$ and the inversion set of w is the disjoint union $N(w) = N(y) \sqcup y^{-1}N(v)$.*

We can now prove that wB is in a parabolic Hessenberg variety $\mathcal{B}(X, \mathfrak{p}_J)$ if and only if the coset representative of w in W^J is in the Springer fiber \mathcal{B}^X . Note that the conditions on X for this result are much less restrictive than those in Section 2.

Proposition 3.4. *Assume that $X \in \mathfrak{g}$ can be written $X = \sum_{\gamma \in \Phi_X} E_\gamma$ for some subset $\Phi_X \subseteq \Phi^+$. Decompose $w = vy$ with $v \in W^J$ and $y \in W_J$. Then $wB \in \mathcal{B}(X, \mathfrak{p}_J)$ if and only if $vB \in \mathcal{B}^X$.*

Proof. By definition the flag $wB \in \mathcal{B}(X, \mathfrak{p}_J)$ if and only if $w^{-1} \cdot X \in \mathfrak{p}_J$ and the flag $vB \in \mathcal{B}^X$ if and only if $v^{-1} \cdot X \in \mathfrak{b}$.

First assume that $wB \in \mathcal{B}(X, \mathfrak{p}_J)$. Consider $v^{-1} \cdot X = \sum_{\gamma \in \Phi_X} E_{v^{-1}(\gamma)}$ and assume the claim fails. Then for some $\gamma \in \Phi_X$ we have $E_{v^{-1}(\gamma)} \notin \mathfrak{b}$ or equivalently $v^{-1}(\gamma) \in \Phi^-$. By Remark 3.1 this implies $v^{-1}(\gamma) \in \Phi^- - \Phi_J^-$. Applying y^{-1} and using Lemma 3.2 gives

$$y^{-1}v^{-1}(\gamma) \in y^{-1}(\Phi^- - \Phi_J^-) = \Phi^- - \Phi_J^-$$

Therefore $y^{-1}v^{-1}(\gamma) \notin \Phi_J^-$ and so $y^{-1}v^{-1} \cdot X \notin \mathfrak{p}_J$ which contradicts the hypothesis on wB .

Now suppose $v^{-1} \cdot X \in \mathfrak{b}$ or equivalently $v^{-1}(\gamma) \in \Phi^+$ for all $\gamma \in \Phi_X$. Either $v^{-1}(\gamma) \in \Phi_J^+$ or $v^{-1}(\gamma) \in \Phi^+ - \Phi_J^+$ for each γ . If $v^{-1}(\gamma) \in \Phi_J^+$ then $y^{-1}v^{-1}(\gamma) \in \Phi_J$ since $y \in W_J$. Otherwise

$$y^{-1}v^{-1}(\gamma) \in y^{-1}(\Phi^+ - \Phi_J^+) = \Phi^+ - \Phi_J^+$$

by Lemma 3.2. Therefore $w^{-1} \cdot X \in \mathfrak{p}_J$ as desired. \square

This proposition implies that if $v \in W^J$ corresponds to a flag vB in the Springer fiber \mathcal{B}^X then $vyB \in \mathcal{B}(X, \mathfrak{p}_J)$ for all $y \in W_J$. That makes the subset $W(X, J) = \{v \in W^J : vB \in \mathcal{B}^X\}$ particularly important in what follows.

Example 3.5. *Let $X \in \mathfrak{gl}_4(\mathbb{C})$ be a nilpotent element in the conjugacy class associated to $(2, 2)$. If X is in highest form as in Definition 2.13 then*

$$X = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $\Phi_X = \{\alpha_1 + \alpha_2, \alpha_2 + \alpha_3\}$. If $J = \{\alpha_1, \alpha_3\}$ then $W^J = \{e, s_2, s_1s_2, s_1s_3s_2, s_2s_1s_3s_2\}$. We find $W(X, J) = \{e, s_2, s_1s_3s_2\}$ by checking whether $v^{-1} \cdot X$ is upper-triangular for each $v \in W^J$.

Now that we have identified the points of the form wB in $\mathcal{B}(X, \mathfrak{p}_J)$ we refine the dimension formula given in Lemma 2.11.

Theorem 3.6. *Fix $J \subseteq \Delta$ and fix an element $X \in \mathfrak{g}$ whose U -orbit is given by Equation (2.9). Let $w \in W$ and write $w = vy$ with $v \in W^J$ and $y \in W_J$. If $wB \in \mathcal{B}(X, \mathfrak{p}_J)$ then*

$$\dim(C_w \cap \mathcal{B}(X, \mathfrak{p}_J)) = \dim(C_v \cap \mathcal{B}^X) + \ell(y).$$

Proof. Lemma 2.11 showed that $wB \in \mathcal{B}(X, \mathfrak{p}_J)$ if and only if the intersection $C_w \cap \mathcal{B}(X, \mathfrak{p}_J)$ is nonempty. We now use the formula in Lemma 2.11 to compute the dimension of the cell $C_w \cap \mathcal{B}(X, \mathfrak{p}_J)$.

The factorization $w = vy$ implies that $w^{-1} = y^{-1}v^{-1}$ and satisfies $\ell(w^{-1}) = \ell(v^{-1}) + \ell(y^{-1})$ so by Lemma 3.3 we have

$$N(w^{-1}) = N(v^{-1}) \sqcup vN(y^{-1}).$$

Thus we can expand the dimension formula from Lemma 2.11 to

$$\begin{aligned} \dim(C_w \cap \mathcal{B}(X, H)) &= |N(v^{-1}) \cap \Phi(\mathcal{V})^c| + |vN(y^{-1}) \cap \Phi(\mathcal{V})^c| \\ &\quad + |N(v^{-1}) \cap \Phi(\mathcal{V}) \cap w(\Phi_J^-)| + |vN(y^{-1}) \cap \Phi(\mathcal{V}) \cap w(\Phi_J^-)|. \end{aligned}$$

Since $w = vy$ we have $N(v^{-1}) \cap w(\Phi_J^-) = N(v^{-1}) \cap vy(\Phi_J^-)$. This intersection is nonempty if and only if there exists a positive root $\gamma \in N(v^{-1})$ such that $y^{-1}v^{-1}(\gamma) \in \Phi_J^-$. The negative root $v^{-1}(\gamma)$ must be in $\Phi^- - \Phi_J^-$ by Remark 3.1. Applying Lemma 3.2 shows $y^{-1}v^{-1}(\gamma) \in y^{-1}(\Phi^- - \Phi_J^-) = \Phi^- - \Phi_J^-$. Therefore $N(v^{-1}) \cap vy(\Phi_J^-)$ is empty for all $y \in W_J$ and the third term in the sum above is zero.

Next $N(y^{-1}) \cap y(\Phi_J^-) = N(y^{-1})$ since $y \in W_J$. Therefore

$$vN(y^{-1}) = v(N(y^{-1}) \cap y(\Phi_J^-)) = vN(y^{-1}) \cap w(\Phi_J^-)$$

for all $v \in W^J$. Thus we can simplify the last term of the dimension formula to obtain

$$\begin{aligned} \dim(C_w \cap \mathcal{B}(X, H)) &= |N(v^{-1}) \cap \Phi(\mathcal{V})^c| + |vN(y^{-1}) \cap \Phi(\mathcal{V})^c| + |vN(y^{-1}) \cap \Phi(\mathcal{V})| \\ &= |N(v^{-1}) \cap \Phi(\mathcal{V})^c| + |vN(y^{-1})|. \end{aligned}$$

Proposition 3.4 showed that $wB \in \mathcal{B}(X, \mathfrak{p}_J)$ if and only if $vB \in \mathcal{B}^X$ and the first term is the dimension of $C_v \cap \mathcal{B}^X$ by Lemma 2.11 as $\Phi_H^- = \emptyset$ in this case. Since $|vN(y^{-1})| = |N(y^{-1})| = \ell(y)$ we obtain

$$\dim(C_w \cap \mathcal{B}(X, H)) = |N(v^{-1}) \cap \Phi(\mathcal{V})^c| + \ell(y) = \dim(C_v \cap \mathcal{B}^X) + \ell(y)$$

as desired. \square

This theorem is the key step in our main result that the Poincaré polynomial of a parabolic Hessenberg variety is a shifted sum of Poincaré polynomials of the flag variety \mathcal{B}_J . The following corollary gives the proof.

Corollary 3.7. *Fix $J \subseteq \Delta$ and $X \in \mathfrak{g}$ such that Equation (2.9) holds. Then*

$$P(\mathcal{B}(X, \mathfrak{p}_J), t) = \sum_{v \in W(X, J)} t^{\dim(C_v \cap \mathcal{B}^X)} P(\mathcal{B}_J, t)$$

where $P(\mathcal{B}(X, \mathfrak{p}_J), t)$ and $P(\mathcal{B}_J, t)$ denote the Poincaré polynomials in variable t of $\mathcal{B}(X, \mathfrak{p}_J)$ and \mathcal{B}_J respectively.

Proof. Applying Lemma 2.6 and Theorem 3.6 we have

$$\begin{aligned} P(\mathcal{B}(X, \mathfrak{p}_J), t) &= \sum_{\{w \in W : wB \in \mathcal{B}(X, \mathfrak{p}_J)\}} t^{\dim(C_w \cap \mathcal{B}(X, \mathfrak{p}_J))} \\ &= \sum_{v \in W(X, J)} \sum_{y \in W_J} t^{\dim(C_v \cap \mathcal{B}^X)} t^{\ell(y)} \\ &= \sum_{v \in W(X, J)} \left(t^{\dim(C_v \cap \mathcal{B}^X)} \sum_{y \in W_J} t^{\ell(y)} \right) \\ &= \sum_{v \in W(X, J)} t^{\dim(C_v \cap \mathcal{B}^X)} P(\mathcal{B}_J, t) \end{aligned}$$

as claimed. \square

Note that the sum $\sum_{v \in W(X, J)} t^{\dim(C_v \cap \mathcal{B}^X)}$ is almost never the Poincaré polynomial of the Springer fiber \mathcal{B}^X because $W(X, J)$ generally does not contain all of the flags $vB \in \mathcal{B}^X$. However in the next section we show that in type A the Betti numbers of $\mathcal{B}(X, \mathfrak{p}_J)$ match those of a union of Schubert varieties for all $X \in \mathfrak{gl}_n(\mathbb{C})$ with Jordan form corresponding to a partition with at most three row or two columns.

Before moving on we give a small example.

Example 3.8. *Let X , J , and $W(X, J)$ be as in Example 3.5. If $v \in W(X, J)$ then $vB \in \mathcal{B}(X, \mathfrak{p}_J)$ by Proposition 3.4. We now use Theorem 3.6 and Lemma 2.11 to calculate $\dim(C_v \cap \mathcal{B}(X, \mathfrak{p}_J))$. We have*

$$\begin{aligned} \dim(C_e \cap \mathcal{B}(X, \mathfrak{p}_J)) &= \dim(C_e \cap \mathcal{B}^X) = |N(e) \cap \Phi(\mathcal{V})^c| = |\emptyset| = 0 \\ \dim(C_{s_2} \cap \mathcal{B}(X, \mathfrak{p}_J)) &= \dim(C_{s_2} \cap \mathcal{B}^X) = |N(s_2) \cap \Phi(\mathcal{V})^c| = |\{\alpha_2\}| = 1 \\ \dim(C_{s_1 s_3 s_2} \cap \mathcal{B}(X, \mathfrak{p}_J)) &= \dim(C_{s_1 s_3 s_2} \cap \mathcal{B}^X) = |N(s_2 s_3 s_1) \cap \Phi(\mathcal{V})^c| = |\{\alpha_1, \alpha_3\}| = 2 \end{aligned}$$

Since we know $W_J = \{e, s_1, s_3, s_1 s_3\}$ Corollary 3.7 now gives the Poincaré polynomial of $\mathcal{B}(X, \mathfrak{p}_J)$:

$$P(\mathcal{B}(X, \mathfrak{p}_J), t) = (1 + t + t^2)(1 + 2t + t^2) = 1 + 3t + 4t^2 + 3t^3 + t^4.$$

Note that this matches the Poincaré polynomial of $\overline{C}_{s_1 s_2 s_3 s_1}$ given in Example 2.7 so the Betti numbers of these two different varieties are the same.

4. SCHUBERT POINTS

The main theorem of this section is that the Betti numbers of parabolic Hessenberg varieties match those of specific unions of Schubert varieties for all $X \in \mathfrak{gl}_n(\mathbb{C})$ with Jordan form corresponding to a partition with at most three row or two columns. More precisely, we will associate to each flag $wB \in \mathcal{B}(X, \mathfrak{p}_J)$ a permutation w_T whose length is the dimension $\dim(C_w \cap \mathcal{B}^X)$ of the Hessenberg Schubert cell for wB . We call w_T the Schubert point corresponding to w .

Our strategy is to show that the map $w \mapsto w_T$ preserves the set W^J . Recall that $W(X, J)$ is the collection of minimal length coset representatives for W/W_J such that the corresponding permutation flag is an element of the Springer fiber. We proved in Section 3 that the flag wB is in the parabolic Hessenberg variety $\mathcal{B}(X, \mathfrak{p}_J)$ if and only if $w = vy$ where $v \in W(X, J)$ and $y \in W_J$. The length $\ell(v_T)$ of the Schubert point corresponding to v is in fact the dimension of the Springer Schubert cell $C_v \cap \mathcal{B}^X$. Therefore by Theorem 3.6, if $v_T \in W^J$ we know that $\ell(v_T y) = \dim(C_w \cap \mathcal{B}(X, \mathfrak{p}_J))$. In other words the length of the permutation $w_T = v_T y$ is the same as the dimension of the parabolic Hessenberg Schubert cell for w so long as $v_T \in W^J$. This gives us the specific union of Schubert varieties that we compare to $\mathcal{B}(X, \mathfrak{p}_J)$. In Theorem 4.10 we prove that the Betti numbers of $\mathcal{B}(X, \mathfrak{p}_J)$ match those of

$$\bigcup_{v \in W(X, J)} \overline{C}_{v_T y_J}$$

where $y_J \in W_J$ denotes the longest word in W_J for all $X \in \mathfrak{gl}_n(\mathbb{C})$ with Jordan form corresponding to a partition with at most three row or two columns.

We assume throughout this section that $X \in \mathfrak{gl}_n(\mathbb{C})$ is in highest form as given in Definition 2.13. (We lose no generality with this assumption by Remark 2.8.) We now describe the points $wB \in \mathcal{B}^X$ combinatorially in terms of row-strict tableaux, namely tableaux whose entries increase from left to right in each row [T, Theorem 7.1].

Lemma 4.1 (Tymoczko). *The permutation flag $wB \in \mathcal{B}^X$ if and only if the tableau T of shape λ given by labeling the i -th box in the base filling of Definition 2.13 by $w^{-1}(i)$ is a row-strict tableau.*

For example the identity permutation corresponds to the base filling of λ . Also note that if i labels a box in T , the corresponding box in the base filling of λ is labeled by $w(i)$.

Not only do the row-strict tableaux index the Springer Schubert cells $C_w \cap \mathcal{B}^X$ but they encode the dimensions $\dim(C_w \cap \mathcal{B}^X)$ as we now describe.

Let T be a row-strict tableau and $T[i]$ be the diagram obtained by restricting T to the boxes labeled $1, \dots, i$. Since T is row-strict, the diagram $T[i]$ consists of rows of boxes and has no gaps—in other words if a box is deleted, all boxes in the same row and to the right must also have been deleted. The following lemma tells how to compute the dimension of the corresponding Springer Schubert cell by counting certain inversions in the tableau T .

Lemma 4.2. *Let $2 \leq q \leq n$ and ℓ_{q-1} be the sum of*

- *the number of rows in $T[q]$ above the row containing q and of the same length, plus*
- *the total number of rows in $T[q]$ of strictly greater length than the row containing q .*

Then

$$\dim(C_w \cap \mathcal{B}^X) = \sum_{i=2}^n \ell_{i-1}$$

We call ℓ_{q-1} the number of q -row inversions of the diagram T .

Lemma 4.2 is an amalgamation of several results. Springer dimension pairs are a subset of the inversions in a filled tableau; the total number of Springer dimension pairs is equal to $\dim(C_w \cap \mathcal{B}^X)$ by work of the second author [T, Theorem 7.1]. The quantities ℓ_{q-1} count the number of Springer dimension pairs of the form (p, q) for $1 \leq p < q \leq n$ and so the sum of the ℓ_{q-1} also gives the total number of Springer dimension pairs [PT, Lemma 2.7].

We next describe a canonical factorization of W following Björner-Brenti's presentation [BB, Corollary 2.4.6]. Recall that the roots associated to the i^{th} row of an upper-triangular matrix are

$$\Phi_i = \{\alpha_i, \alpha_i + \alpha_{i+1}, \dots, \alpha_i + \alpha_{i+1} + \dots + \alpha_{n-1}\} \text{ for each } 1 \leq i \leq n-1.$$

Lemma 4.3 (Björner-Brenti). *Each $w \in W$ can be written uniquely as $w = w_{n-1}w_{n-2} \dots w_2w_1$ where*

$$w_i = s_{k_i}s_{k_i+1} \dots s_{i-1}s_i \text{ for each } i = 1, \dots, n-1$$

and either $w_i = e$ or k_i is a fixed integer with $1 \leq k_i \leq i$. We call w_i the i -th string of w . Moreover

$$w_1^{-1}w_2^{-1} \dots w_{i-1}^{-1}N(w_i) \subseteq \Phi_i \text{ for each } i = 1, \dots, n-1.$$

For example the longest word in S_4 can be written as $s_1s_2s_3s_1s_2s_1$. In this case the strings are

- $w_3 = s_1s_2s_3$
- $w_2 = s_1s_2$ and
- $w_1 = s_1$

so $k_i = 1$ for all $i = 1, 2, 3$. Note that if $w_i \neq e$ then $\ell(w_i) = i - k_i + 1$ in general.

In previous work we studied a bijection between $wB \in \mathcal{B}^X$ and certain permutations $w_T \in W$ whose lengths are the dimension of the corresponding Springer Schubert cells [PT, Definition 3.2].

Definition 4.4 (Schubert points). *Let $wB \in \mathcal{B}^X$ and let T denote the corresponding row-strict tableau. For each $2 \leq q \leq n$ let ℓ_{q-1} be the number of q -row inversions of T . Define a string w_{q-1} by*

$$w_{q-1} = \begin{cases} s_{q-\ell_{q-1}}s_{q-\ell_{q-1}+1} \dots s_{q-2}s_{q-1} & \text{if } \ell_{q-1} \neq 0 \\ e & \text{if } \ell_{q-1} = 0 \end{cases}$$

so w_{q-1} is a string of length ℓ_{q-1} by construction. Then

$$w_T = w_{n-1}w_{n-2} \dots w_2w_1$$

is the Schubert point associated to $wB \in \mathcal{B}^X$.

By construction

$$\ell(w_T) = \ell_{n-1} + \ell_{n-2} + \cdots + \ell_1 = \dim(C_w \cap \mathcal{B}^X).$$

In fact not only are the words w_T in bijection with row-strict tableaux, but the set $\{w_T : T \text{ is row strict}\}$ forms a lower order ideal in the Bruhat graph whenever λ has at most three rows or two columns—namely the elements of the set index a union of Schubert varieties [PT, Theorem 4.4].

Lemma 4.5 (Precup-Tymoczko). *For each $wB \in \mathcal{B}^X$ there exists a unique Schubert point $w_T \in W$. In addition, if the Jordan form of X corresponds to a partition with at most three rows or two columns then every permutation $w' \leq w_T$ in Bruhat order corresponds to a unique $yB \in \mathcal{B}^X$ such that $w' = y_{T'}$ for the row-strict tableau T' corresponding to yB .*

Our plan to extend this result is to show that the Schubert points respect the decomposition $W^J W_J$. More precisely we will show that $v \in W^J$ if and only if the Schubert point v_T corresponding to v is an element of W^J . We begin with an alternate characterization of W^J .

Proposition 4.6. *Let $w \in W$ and write $w = w_{n-1}w_{n-2} \cdots w_2w_1$ where w_i denotes the i -th string of w for each $i = 1, 2, \dots, n-1$. Then $w \in W^J$ if and only if $\ell(w_i) \leq \ell(w_{i-1})$ for all $\alpha_i \in J$.*

Proof. We will prove the contrapositive statement using Remark 3.1, which says that w is not in W^J if and only if there is a simple root $\alpha_i \in J$ for which $\alpha_i \in N(w)$. In particular we prove that for each simple root $\alpha_i \in J$, the root $\alpha_i \in N(w)$ if and only if $\ell(w_i) > \ell(w_{i-1})$.

Since $\ell(w) = \ell(w_{n-1}) + \ell(w_{n-2}) + \cdots + \ell(w_2) + \ell(w_1)$ we can write

$$N(w) = N(w_1) \sqcup w_1^{-1}N(w_2) \sqcup \cdots \sqcup w_1^{-1}w_2^{-1} \cdots w_{n-2}^{-1}N(w_{n-1})$$

by Lemma 3.3. Given $\alpha_i \in J$ consider $w_i = s_{k_i}s_{k_i+1} \cdots s_{i-1}s_i$ and $w_{i-1} = s_{k_{i-1}}s_{k_{i-1}+1} \cdots s_{i-2}s_{i-1}$. Note that

$$(4.7) \quad N(w_i) = \{\alpha_i, s_i(\alpha_{i-1}), \dots, s_i s_{i-1} \cdots s_{k_i+1}(\alpha_{k_i})\}.$$

By Lemma 4.3 we know $\alpha_i \in N(w)$ if and only if $\alpha_i \in w_1^{-1}w_2^{-1} \cdots w_{i-2}^{-1}w_{i-1}^{-1}N(w_i)$. Since $\ell(w_i) = i - k_i + 1$ we know

$$\ell(w_i) > \ell(w_{i-1}) \quad \text{if and only if} \quad i - k_i + 1 > i - 1 - k_{i-1} + 1$$

This in turn is equivalent to $k_i \leq k_{i-1}$ and implies that the reflection $s_{k_{i-1}}$ must occur in the word $w_i = s_{k_i}s_{k_i+1} \cdots s_{i-1}s_i$. The description of $N(w_i)$ in Equation (4.7) shows that this is the case if and only if

$$s_i s_{i-1} \cdots s_{k_{i-1}+1}(\alpha_{k_{i-1}}) = \alpha_{k_{i-1}} + \alpha_{k_{i-1}+1} + \cdots + \alpha_{i-1} + \alpha_i \in N(w_i).$$

Thus $k_i \leq k_{i-1}$ if and only if

$$w_1^{-1}w_2^{-1} \cdots w_{i-2}^{-1}w_{i-1}^{-1}(\alpha_{k_{i-1}} + \alpha_{k_{i-1}+1} + \cdots + \alpha_{i-1} + \alpha_i) \in N(w)$$

But

$$\begin{aligned} w_{i-1}^{-1}(\alpha_{k_{i-1}} + \alpha_{k_{i-1}+1} + \cdots + \alpha_{i-1} + \alpha_i) &= \\ s_{i-1}s_{i-2} \cdots s_{k_{i-1}+1}s_{k_{i-1}}(\alpha_{k_{i-1}} + \alpha_{k_{i-1}+1} + \cdots + \alpha_{i-1} + \alpha_i) &= \alpha_i \end{aligned}$$

and w_1, w_2, \dots, w_{i-2} stabilize α_i . Putting this together, we conclude $\ell(w_i) > \ell(w_{i-1})$ if and only if $\alpha_i \in N(w)$ as desired. \square

The previous lemma is the key step in the next proposition, which shows that if $v \in W^J$ indexes a permutation flag $vB \in \mathcal{B}^X$ then the corresponding Schubert point v_T is also in W^J .

Proposition 4.8. *Let $vB \in \mathcal{B}^X$. Then $v \in W^J$ if and only if $v_T \in W^J$.*

Proof. Let T denote the row-strict tableau associated to v . We decompose v_T into i -strings as $v_T = v_{n-1}v_{n-2} \cdots v_2v_1$. Throughout this proof, assume i satisfies $1 \leq i \leq n-1$ and $\alpha_i \in J$.

By definition $\ell(v_i) = \ell_i$ and $\ell(v_{i-1}) = \ell_{i-1}$ so by Proposition 4.6 and Remark 3.1 we have only to show that $\alpha_i \notin N(v)$ if and only if $\ell_i \leq \ell_{i-1}$. First $\alpha_i \notin N(v)$ if and only if $v(i) < v(i+1)$. Since i fills the box labeled by $v(i)$ in the base filling of λ , the inequality $v(i) < v(i+1)$ holds if and only if i occurs in a box of T

- in the same column and below $i+1$, or
- in a column to the left of $i+1$.

Now consider $T[i]$ and $T[i+1]$. We obtain $T[i]$ from $T[i+1]$ by removing the box containing $i+1$. Lemma 4.2 states that ℓ_i counts the number of rows in $T[i+1]$ above the row containing $i+1$ and of equal length plus the total number of rows in $T[i+1]$ of length strictly greater than the row with $i+1$. These rows each have the same length in $T[i]$ since they do not contain $i+1$; denote the set of rows by \mathcal{R} . If i satisfies either bulleted condition above then each row in \mathcal{R} contributes one i -row inversion of T to the count of ℓ_{i-1} so by Lemma 4.2 we have $\ell_i = |\mathcal{R}| \leq \ell_{i-1}$. Conversely if i satisfies neither bulleted condition then ℓ_{i-1} counts only a subset of \mathcal{R} since \mathcal{R} includes the row containing i . Therefore $\ell_{i-1} < |\mathcal{R}| = \ell_i$. This proves the claim. \square

Corollary 4.9. *Suppose X corresponds to a partition with at most three rows or two columns. Then the set $\{v_T \in W^J : v \in W^J \text{ and } vB \in \mathcal{B}^X\}$ is a lower order ideal with respect to Bruhat order on W^J . In other words if $v' \in W^J$ and $v' \leq v_T$ for some v_T in the set, then v' is also an element of the set.*

Proof. To prove this, we show that for each $v' \in W^J$ such that $v' \leq v_T$ there exists $y \in W^J$ with $yB \in \mathcal{B}^X$ and row-strict tableau T' such that $v' = y_{T'}$. By Proposition 4.5, there exists a unique $yB \in \mathcal{B}^X$ and corresponding row-strict tableau T' such that $v' = y_{T'}$. By Proposition 4.8 this y must also be an element of W^J since $y_{T'}$ is. \square

We are now ready to state and prove the main theorem.

Theorem 4.10. *Suppose $X \in \mathfrak{gl}_n(\mathbb{C})$ is a nilpotent element with Jordan form corresponding to a partition with at most three rows or two columns. Then the following Poincaré polynomials are equal:*

$$P(\mathcal{B}(X, \mathfrak{p}_J), t) = P(\cup_{v \in W(X, J)} \overline{C}_{v_T w_J}, t)$$

where w_J denotes the longest word in W_J .

Proof. We know that the union of Schubert varieties is the disjoint union of Schubert cells

$$\bigcup_{v \in W(X, J)} \overline{C}_{v_T w_J} = \bigsqcup_{v \in W(X, J)} \bigsqcup_{y \in W_J} C_{v_T y}$$

because $W(X, J)$ is a subset of coset representatives for W/W_J . Recall that \mathcal{B}_J denotes the flag variety $M_J/(B \cap M_J)$. Then we get

$$\begin{aligned} P(\cup_{v \in W(X, J)} \overline{C}_{v_T w_J}, t) &= \sum_{v \in W(X, J)} t^{\ell(v_T)} P(\mathcal{B}_J, t) \\ &= \sum_{v \in W(X, J)} t^{\dim(C_v \cap \mathcal{B}^X)} P(\mathcal{B}_J, t) \\ &= P(\mathcal{B}(X, \mathfrak{p}_J), t) \end{aligned}$$

where the last two equalities follow by Definition 4.4 and Corollary 3.7, respectively. \square

5. COMPONENTS OF PARABOLIC HESSENBERG VARIETIES

The natural follow-up question is whether the combinatorial results in Proposition 4.8 and Theorem 4.10 reflect an underlying geometric property. We now give one result in this direction that partially characterizes the irreducible components of parabolic Hessenberg varieties. It extends results about Springer fibers in type A , whose irreducible components are known to be indexed by standard tableaux.

The result proves that if v is any element of W^J and $y \leq y'$ are two elements of W_J then the Hessenberg Schubert cell corresponding to vy lies in the closure of the Hessenberg Schubert cell corresponding to vy' . The proof depends on Proposition 3.4, which related the structure of a parabolic Hessenberg variety to the elements in the set $W(X, J)$, and also on techniques of Bernstein-Gelfand-Gelfand to compute the closure relations between Schubert cells [BGG]. The proposition holds in all Lie types.

Proposition 5.1. *Let $v \in W^J$ such that $vB \in \mathcal{B}^X$, let w_J denote the longest word in W_J , and $X \in \mathfrak{g}$ be a nilpotent element. Then $C_{vy} \cap \mathcal{B}(X, \mathfrak{p}_J) \subseteq \overline{C_{vw_J} \cap \mathcal{B}(X, \mathfrak{p}_J)}$ for all $y \in W_J$.*

Proof. Fix any pair $y_1, y_2 \in W_J$ such that $y_1 = s_\gamma y_2$ for some reflection s_γ . Assume without loss of generality that $y_1 \leq y_2$. The reflection s_γ must be an element of W_J and so $\gamma \in \Phi_J$. Let $u_\gamma(t) = \exp(tE_\gamma)$ for each $t \in \mathbb{C}$. Note that $u_\gamma(t)$ is a unipotent element of U . The closure in the flag variety is

$$\lim_{t \rightarrow \infty} u_\gamma(t)y_2B = y_1B$$

by work of Bernstein-Gelfand-Gelfand [BGG, Theorem 2.11]. Every flag in $C_{vy_1} \cap \mathcal{B}(X, \mathfrak{p}_J)$ is of the form uvy_1B for some $u \in U$. For each such flag we have

$$u^{-1} \cdot X \in vy_1 \cdot \mathfrak{p}_J$$

by definition of Hessenberg varieties. Since \mathfrak{p}_J is preserved as a set both by U and W_J we have

$$vy_1 \cdot \mathfrak{p}_J = v \cdot \mathfrak{p}_J = vu_\gamma(t) \cdot \mathfrak{p}_J = vu_\gamma(t)y_2 \cdot \mathfrak{p}_J$$

Thus $uvu_\gamma(t)y_2B \in C_{vy_2} \cap \mathcal{B}(X, \mathfrak{p}_J)$. It follows that every element of $C_{vy_1} \cap \mathcal{B}(X, \mathfrak{p}_J)$ is in the closure of $C_{vy_2} \cap \mathcal{B}(X, \mathfrak{p}_J)$ since

$$\lim_{t \rightarrow \infty} uvu_\gamma(t)y_2B = uv y_1B$$

for all $uvy_1B \in C_{vy_1} \cap \mathcal{B}(X, \mathfrak{p}_J)$. The claim now follows by induction on the length of $y \in W_J$. \square

Consider the following example, which shows that the irreducible components of parabolic Hessenberg varieties need not be indexed by standard tableaux as they are in the case of Springer fibers.

Example 5.2. *Let X correspond to the orbit indexed by the partition $(2, 1, 1)$ in $\mathfrak{sl}_4(\mathbb{C})$ and the parabolic subalgebra be determined by $J = \{\alpha_1, \alpha_3\}$ so $w_J = s_1s_3$. Then $\Phi_X = \{\alpha_3\}$, $\Phi(\mathcal{V}) = \{\alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$, and one can show that we get the following elements of $W(X, J)$,*

$$W(X, J) = \{e, s_2, s_1s_2, s_2s_1s_3s_2\}.$$

By Proposition 5.1, the irreducible components are indexed by some subset of $W(X, J)$. We consider the points $v_1 = s_1s_2$ and $v_2 = s_2s_1s_3s_2s_2$. These elements are listed below with their corresponding row-strict tableau.

$v \in W(X, J)$	row-strict tableau	v_T	$v_T w_J$
$v_1 = s_1 s_2$	$\begin{array}{ c c } \hline 2 & 4 \\ \hline \end{array}$	$s_1 s_2$	$s_1 s_2 s_1 s_3$
	$\begin{array}{ c } \hline 1 \\ \hline \end{array}$		
	$\begin{array}{ c } \hline 3 \\ \hline \end{array}$		
$v_2 = s_2 s_1 s_3 s_2$	$\begin{array}{ c c } \hline 1 & 2 \\ \hline \end{array}$	$s_3 s_2$	$s_3 s_2 s_1 s_3$
	$\begin{array}{ c } \hline 4 \\ \hline \end{array}$		
	$\begin{array}{ c } \hline 3 \\ \hline \end{array}$		

First we claim that $\overline{C_{v_1 w_J} \cap \mathcal{B}(X, \mathfrak{p}_J)}$ and $\overline{C_{v_2 w_J} \cap \mathcal{B}(X, \mathfrak{p}_J)}$ are the irreducible components of $\mathcal{B}(X, \mathfrak{p}_J)$. Since $\dim(C_{v_1 w_J} \cap \mathcal{B}(X, \mathfrak{p}_J)) = \dim(C_{v_1 w_J})$ it follows that $C_{v_1 w_J} \cap \mathcal{B}(X, \mathfrak{p}_J) = C_{v_1 w_J}$ and therefore

$$\overline{C_{v_1 w_J} \cap \mathcal{B}(X, \mathfrak{p}_J)} = \overline{C_{v_1 w_J}} \cap \mathcal{B}(X, \mathfrak{p}_J) = \bigsqcup_{w \leq v_1 w_J} C_w \cap \mathcal{B}(X, \mathfrak{p}_J).$$

Since $v_2 w_J \not\leq v_1 w_J$ and $vw_J \leq v_1 w_J$ for all other $v \in W(X, J)$ we obtain

$$\mathcal{B}(X, \mathfrak{p}_J) = \overline{C_{v_1 w_J}} \cup (\overline{C_{v_2 w_J} \cap \mathcal{B}(X, \mathfrak{p}_J)}).$$

Note in particular that not every standard tableau of shape $(2, 1, 1)$ indexes an irreducible component of a parabolic Hessenberg variety—and not every component is indexed by a standard tableau. As in this example, the indexing diagram may be a row-strict tableau. It is also known that parabolic Hessenberg varieties are not in general equidimensional [T3]. Since it is well known that Springer varieties are equidimensional [S], this is one way in which their geometry differs.

Our partial description of the irreducible components of $\mathcal{B}(X, \mathfrak{p}_J)$ lead to the following question.

Question 5.3. Give a combinatorial description of those $v \in W(X, J)$ that index an irreducible component of $\mathcal{B}(X, \mathfrak{p}_J)$.

Motivated by Example 5.2, one possible description is that $\overline{C_{vw_J} \cap \mathcal{B}(X, \mathfrak{p}_J)}$ is an irreducible component of $\mathcal{B}(X, \mathfrak{p}_J)$ if the Schubert point $v_T w_J$ corresponding to vw_J is a maximal element of $\{v_T w_J : v \in W(X, J)\}$. We have not been able to find counterexamples to this description, but suspect that there is one.

An answer to Question 5.3 would extend the known characterization of the components of the Springer fiber of type A and also seems to require a deep analysis of the set $W(X, J)$.

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